Guidance and Trajectory Considerations in Lunar Mass Transportation

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An introductory discussion of flight-mechanics problems associated with large-scale transport of lunar mass to a space colony or manufacturing facility is presented. The proposed transport method involves launch of payloads from a mass-driver on the lunar surface, onto ballistic trajectories to a passive mass-catcher located near the L_2 libration point, with the caught mass subsequently being transported to the colony. We treat arrival velocities at L_2, sensitivities in arrival dispersion due to launch errors, and effects of launch site location, via numerically integrated orbits in the restricted three-body problem. A unifying principle exists, whereby from any launch site it is possible to define a target point which is reached with zero dispersion due to errors in a selected component of launch velocity. We treat the effects of lunar geometrical librations and of obliquity, as well as the conditions for biasing a trajectory away from L_2 so as to reduce stationkeeping costs. We also treat transfer orbits from L_2 to the colony. We show that the theory of capture and the theory of resonance lead to a colony orbit, with period approximately two weeks, reached from L_2 with ΔV as low as 9.02 m/sec.

Introduction

SINCE 1974 there has been increasing interest in the concept, originally due to O'Neill,[1,2] of establishing space colonies or manufacturing facilities, to support construction of solar power satellites and other large space systems. The resources for the construction are to be drawn from the Moon. Thus, the large-scale transport of lunar material is a matter of paramount importance. The overall considerations of space colonization have been treated in preliminary studies[3,4] and the technology for lunar mass transport has been described elsewhere.[5] We here consider problems of guidance and flight mechanics, which will strongly influence further studies of this technology.

The lunar mass transportation system, which has received the greatest attention to date, is illustrated in Fig. 1. The mass-driver applies electric energy to accelerate packages of lunar matter to escape velocity; it has been the subject of feasibility studies.[5,6] The payloads then fly ballistically, without midcourse correction, to a mass-catcher near the L_2 libration point. This catcher is quite poorly defined, but for the purposes of this paper, it is regarded as having two characteristics. It has diameter of 100 meters; and it is equipped with onboard propulsion, such as the Rotary Pellet Launcher system.[5] This propulsion system serves for stationkeeping, for following the mass stream, and for transport to the colony.

The study of lunar mass transport then divides naturally into four problems: Aim sensitivities at the Moon and payload arrival velocities at L_2; counteracting the momentum flux due to the payload stream at the catcher; system operation under effects of perturbations and of lunar libration and obliquity; transfer to the colony. Each of these will be treated in turn.

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Trajectories, Arrival Velocities, and Aim Sensitivities

The principal results, dealing with transfer trajectories from the Moon to L_2, are due to Nicholson,[7] Edelbaum,[8] and D'Amario and Edelbaum.[9] The transfers are of three types. Originating from sites in the lunar eastern hemisphere, for easterly launches, are the "slow" transfers, with transit times of 200-250 hours and arrival velocities at L_2 of 105-115 hours. From this same hemisphere, also launching easterly, are the "fast" transfers, with transit times of 50-100 hours, arrival velocities of 150-250 m/sec. There are also a class of "super-fast" trajectories, with transfer times under 24 hours, arrival velocities above 300 m/sec. These involve westward launches, mostly from sites in the western hemisphere. Of these, the "fast" transfers were studied.

They were studied in the planar restricted three-body problem, being integrated from L_2 to the Moon. In view of the symmetry properties of solutions,[10] trajectories obtained by reflecting solutions across the x-axis (the Earth-Moon line

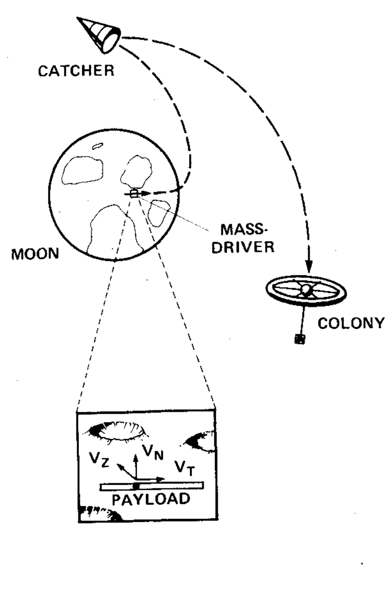

Fig. 1 Elements of the problem of lunar mass transport.

of centers) constitute transfers from the Moon to L_2. Integration of solutions was performed in double precision on the Ames Research Center's CDC 7600 computer. A step size of 0.005 time units was used beyond three lunar radii, 0.0001 time units inside that region. To further improve the accuracy of numerically converged trajectories, the step size was further reached to 0.00001 time units when inside a normalized distance of 0.00455 units from the lunar center; lunar radius was taken as 0.00452121 units. The usual restricted three-body normalizations were employed: lunar mass $\mu = 0.01215$, unit distance = 384,410 km, unit time = 104.362 hours, unit velocity = 1023.17 m/sec, unit acceleration = 0.00273 m/sec^2.

Solutions were initiated using data from Ref. 9, giving approximate values of velocity magnitudes and directions at L_2 for transfer to the Moon. For each trajectory found, these values were used to initialize a Newton iteration wherein the direction at L_2 was held constant while the velocity magnitude was changed. The condition on convergence was that the trajectory be at 1.0 lunar radius at closest approach. Typically, the Jacobi constant was preserved to nine decimal places along the trajectory; the point of closest approach was found as 0.00452133 units (a 40-meter error in physical distance); and the orbit, at closest approach, was within 0.1 to 1.0 milliradian of having the velocity vector tangent to the lunar surface. The equations of motion were taken in a rotating Earth-centered system of coordinates:

$$\ddot{x} - 2\dot{y} = \Omega_x; \qquad \ddot{y} + 2\dot{x} = \Omega_y; \qquad \ddot{z} = \Omega_z \tag{1a}$$

$$\Omega = \tfrac{1}{2}[(x-\mu)^2 + y^2] + (1-\mu)/r_1 + \mu/r_2 \tag{1b}$$

$$r_1^2 = x^2 + y^2 + z^2; \qquad r_2^2 = (x-1)^2 + y^2 + z^2 \tag{1c}$$

The primaries are located at $x=0$, $x=1$; the z-axis is out of plane; the Jacobi constant $C = -(\dot{x}^2 + \dot{y}^2) + 2\Omega$. Nominal solutions are found with $z=0$.

Once the trajectories were found, it was possible to study their sensitivities with respect to launch errors. These were studied by the direct method of perturbing the initial conditions, at launch from the Moon, and observing their effect upon the closest approach distance to L_2. This method was selected in preference to one involving integration of the linear variational equations[11] insofar as the latter method required extensive blocks of computer time and did not lead to a readily observable physical picture of the effect of a perturbation on the orbit. By reducing the time step to 0.0001 units when within 0.01 distance units of L_2, it was possible to read directly from the computer printout a value for the closest approach distance which was generally within 10^{-6} units of the "true" distance.

Figure 1 shows the components of velocity at the moment of launch. V_T is tangent to the surface, along the direction of the mass-driver track. V_N is normal to the surface, while V_Z is to the left or right of the track. Since the track involves one degree of freedom, in a nominal launch there is $V_N = V_Z = 0$. The associated errors in the three components give rise to sensitivity coefficients C_{VT}, C_{VN}, C_{VZ}. These are defined as the miss distance at L_2, in meters, respectively due to an error of 10^{-3} m/sec in V_T, V_N, V_Z. They were found by integrating Eqs. (1) from the Moon to L_2 three times, respectively with errors in V_T, V_N, V_Z of 0.001 velocity units.

In addition, two other sensitivity coefficients were found: C_ϕ and C_Z. These give the miss distance at L_2, in meters, respectively due to an error in position of 1.0 meters along the track (in the direction of increasing longitude ϕ) and to the side of the track. Reflecting the radial geometry of motion near the Moon, both values are found to be somewhat close to 37.1, the distance from the Moon to L_2 in units of a lunar radius.

Results from the computations are presented in Figs. 2-4. Figure 2 depicts several trajectories initially tangent to the lunar surface and passing through L_2. The numbers labelling each curve give the longitude at launch in radians. There is evident similarity of these results to those of Refs. 7 and 9.

Figure 3 gives launch and arrival conditions, as functions of longitude ϕ at launch. The data given are: flight time t_f; launch velocity V_T; arrival velocity V_{L2}; and arrival angle γ. As would be predicted from consideration of the Jacobi constant, the shapes of the curves for V_T and V_{L2} are entirely similar, but $\partial V_{L2}/\partial V_T = 10$, very nearly. This is reflected in the different scales used for V_T, V_{L2}. Angle γ is measured with respect to the x-axis, positive clockwise. These data closely resemble those of Fig. 6 of Ref. 9. The minimum value of V_{L2}, 147.8 m/sec, is identical to that found by Nicholson.[7]

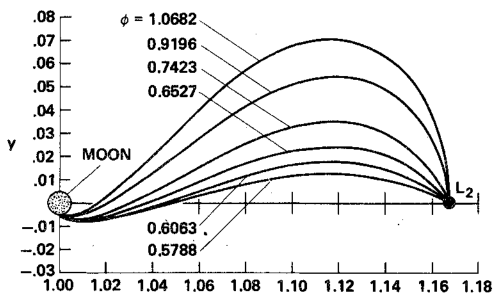

Fig. 2 Trajectories from the Moon to L_2.

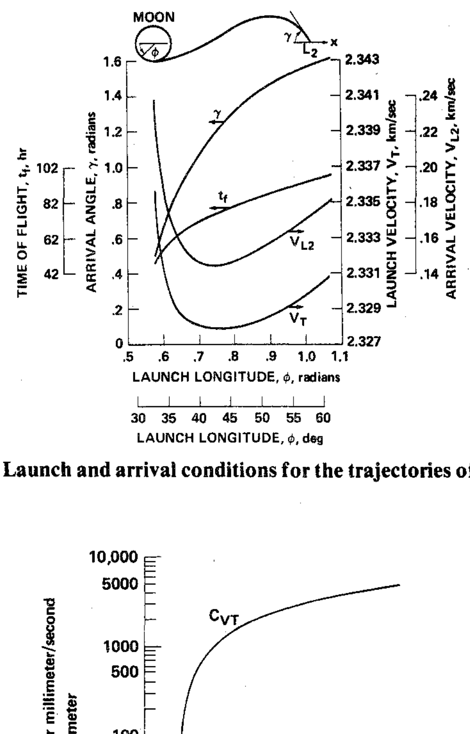

Fig. 3 Launch and arrival conditions for the trajectories of Fig. 2.

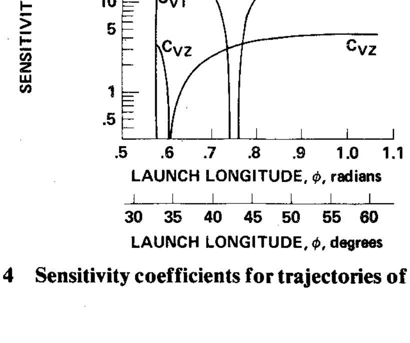

Fig. 4 Sensitivity coefficients for trajectories of Fig. 2.

Figure 4 gives C_ϕ, C_{VT}, C_{VN}, C_{VZ}, C_Z, again as functions of ϕ at launch. A most significant result is seen: that for all three of C_{VT}, C_{VN}, C_Z there exist values of ϕ for which they go to zero. Hence it is possible to launch from such a location and the payload stream at L_2 will suffer zero dispersion, to first order, due to an error in the associated component of launch velocity.

Because of the importance of this result to the problem of lunar mass transport, it is necessary to give a theory for such critical points.

Critical Launch Points

The vanishing of C_{VZ} is recognized as a purely geometrical effect, akin to launching along a trajectory which is an arc of a great circle. In the two-body problem, C_{VZ} would vanish when $\phi = t_f$; the deviation from this condition here reflects the perturbations due to the Earth. The vanishing of C_{VT}, C_{VN}, however, involves more complex phenomena and is most readily treated using methods of function theory or topology. Under this viewpoint, a family of trajectories, initially tangent to the lunar surface and terminating at a point P in space, is regarded as a function or mapping which maps a manifold of initial conditions at the Moon into a manifold of final conditions at P. This topological approach has hitherto found application to such problems as the identification of new integrals associated with motion under generalized potentials.[12]

To gain freedom in investigation, we do not restrict P to be L_2, and we avoid the use of conditions such as the existence of the Jacobi integral. We consider the mapping $F(\phi, V_T) \rightarrow (\gamma, V_P)$, i.e. any trajectory in family F maps initial conditions (ϕ, V_T) at the Moon into final conditions (γ, V_P) at P, where V_P = velocity magnitude. There is also the inverse mapping $F^{-1}(\gamma, V_P) \rightarrow (\phi, V_T)$ the existence of which follows from the time-reversibility of dynamical systems. Both mappings are locally continuous and single-valued.

To begin, note that $V_T \, d\phi/dV_N = 1$, i.e. the presence of $dV_N \neq 0$ shifts the effective launch site from ϕ to $\phi + d\phi$. Therefore we have:

Theorem 1. The conditions that $C_{VN} = 0$ is that ϕ be such that V_P is a local minimum or maximum, and that $V_T = V_T(V_P)$ only.

To prove this, let V_P be a locally extreme value, so that the mappings $F^{-1}(\gamma, V_P) \rightarrow (\phi, V_T)$, $F^{-1}(\gamma + d\gamma, V_P) \rightarrow (\phi + d\phi, V_T + dV_T)$ exist. If $V_T = V_T(V_P)$ then $F^{-1}(\gamma + d\gamma, V_P) \rightarrow (\phi + d\phi, V_T)$ and hence $F(\phi + d\phi, V_T) \rightarrow (\gamma + d\gamma, V_P)$. Now let $d\phi = dV_N / V_T$; then there exists a range of dV_N for which F maps points $(\phi + dV_N/V_T, V_T)$ onto P; which is to say, $C_{VN} = 0$.

The condition $V_T = V_T(V_P)$ is equivalent to the assertion that at the lunar surface, Earth and solar perturbations are sufficiently small that V_T may be treated as due to lunar gravity only. This condition is satisfied to a relative accuracy of 0.8×10^{-9}. The numerical integrations give minimum $V_{L2} = 0.14442656$ for $V_T = 2.27484569$ at $\phi = 0.744305059 = 42°38'43.94''$ east longitude; there, in normalized units, $C_{VN} = 0.00000027 = 0.1$ m/mm/sec.

Now consider the condition that $C_{VT} = 0$. To begin, we assert that given the Moon of radius r_m and a point P in space, beyond r_m, there exists a family of trajectories $F_1^{-1}(\gamma, V_P) \rightarrow (\phi, V_T)$ for which $\partial\phi/\partial V_T \geq 0$. The proof of this is simple: From the geometry of straight lines tangent at the Moon and passing through P, we have $F_1^{-1}(\gamma = \sin^{-1}(r_m/r_p),\ V_P = \infty) \rightarrow (\phi = \pi/2 + \gamma,\ V_T = \infty)$, where r_p = distance of P from the lunar center and ϕ is measured from the line, lunar center to P. Since V_T, $V_P = \infty$, the cited value of ϕ is clearly a maximum. Then, from the continuity of F_1^{-1}, there exists a range of $V_P < \infty$ for which $V_T < \infty$ and for which the image under F_1^{-1} of (γ, V_P) is (ϕ, V_T) with $\phi < \pi/2 + \sin^{-1}(r_m/r_p)$. That is, by reducing the velocity we cause the tangent point ϕ at the Moon to shift in the direction away from point P, so that $\partial\phi/\partial V_T \geq 0$.

We now inquire as to the minimum value of ϕ which is associated with F_1^{-1}, i.e. the minimum ϕ reachable from P. Let this be ϕ_{min}. We then inquire as to the effect of further reducing V_P. In the two-body problem $\phi_{min} = 0°$ (180° away from the sub-P point on the Moon) and further reduction of V_P involves the nonexistence of the mapping F_1^{-1}, i.e. the trajectory cannot be made to pass tangent to the lunar surface. But Fig. 3 illustrates the existence of another family, $F_2(\phi, V_T) \rightarrow (\gamma, V_P)$, for which it is seen that $\partial\phi/\partial V_T \leq 0$. Now suppose that $F_2(\phi_{min}, V_T) = F_1(\phi_{min}, V_T)$ so that ϕ_{min}

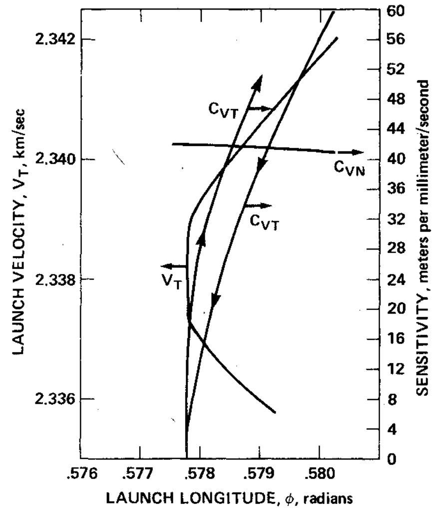

Fig. 5 The branch point or extremum on ϕ.

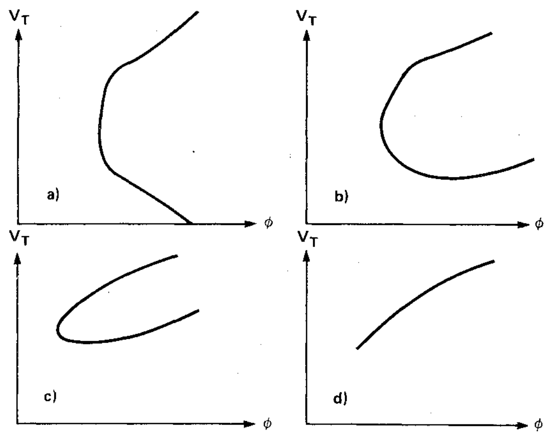

Fig. 6 Schematic evolution of the branch point well away from L_2.

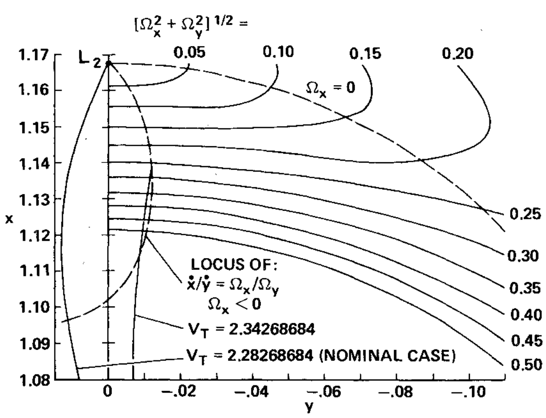

Fig. 7 Contours of potential acceleration and the locus of dynamical libration points.

is a branch point between the two families. It then is necessary both that $\partial\phi/\partial V_T \geq 0$ and $\partial\phi/\partial V_T \leq 0$; hence at ϕ_{min}, $\partial\phi/\partial V_T = 0$. This is equivalent to the statement, $C_{VT}=0$. We have thus proved the following.

Theorem 2. The condition that $C_{VT}=0$ is that ϕ be the branch point joining two families of trajectories to P, one with $\partial\phi/\partial V_T \geq 0$ and the other with $\partial\phi/\partial V_T \leq 0$.

To apply this theorem for engineering purposes, it is sufficient to find one point P for which there exists such a critical ϕ. The continuity of the manifolds (ϕ, V_T) and (γ, V_P) and of the mappings F_1, F_2 and their inverses then guarantees the existence of a nonzero field of points near P for which critical values, ϕ_c, also exist. But this has already been found: P is L_2, and $\phi = 0.577768148 = 33°6'13.23''$ east longitude.

Figure 5 gives detail of V_T, C_{VT}, C_{VN} in the vicinity of this value of ϕ_c. The double-valued curve $V_T = V_T(\phi)$ clearly exhibits the nature of ϕ_c as a branch point between two families of trajectories. The curve for C_{VN} illustrates the range of values of ϕ for which $C_{VT} < C_{VN}$. The curve $C_{VT} = C_{VT}(\phi)$ is double-valued, its two branches corresponding to the two trajectory families; the downward-sloping branch is the limit of Fig. 4. The value for ϕ_c cited is that for which the lowest value of C_{VT} was found numerically: $C_{VT} = 0.000006$ in normalized units, or $C_{VT} = 2.2$ m/mm/sec.

It is of interest how far one may go from L_2 before branch-point values of ϕ fail to exist. In principle, there exist points for which ϕ_{min} is not a branch point but simply a limit point, as in the two-body problem. At such a point, $\partial\phi/\partial V_T \neq 0$ and $C_{VT} \neq 0$. Topologically, as one approaches the limit of the region where $C_{VT}=0$ may hold, the curve $V_T = V_T(\phi)$ evolves as in Fig. 6. Initially the two branches are distinct and well separated. Nea the limit, they grow closer together, the region where $\partial\phi/\partial V_T = 0$ is more sharply defined, and the region wherein $C_{VT} < C_{VN}$ grows narrower. Finally, the branch point becomes a cusp and the curves merge to give a single curve with a limit point.

Edelbaum[8] has noted the close similarity between orbits from the Moon to L_2 and orbits from the Moon to the near vicinity of L_2. Thus, it appears entirely reasonable to propose that critical points ϕ_c exist for target points P up to 0.025 distance units from L_2, which is the range of variation required here. (L_2 is 0.167833 from the Moon.) Accordingly, we propose the following strategy for operation of the mass-driver:

Definition 1. The target $P = P(t)$ for the mass stream, as a function of time, is the locus of points closest to L_2 for which the longitude of the mass-driver constitutes a critical point, at any time t.

The cited equatorial location, at 33.1° east, has the disadvantage of lying in the mountainous region near the craters Maskelyne and Censorinus. But a short distance away, at 1°50′ north, 33°40′ east, is a smooth plain representing the southern extremity of Mare Tranquillitatis. There, a due-east tangential launch is possible, with an unobstructed flight of 250 km before the payloads pass 6000 meters above the saddle of terrain joining the craters Secchi and Lubbock S. This site appears eminently suitable for the mass-driver.[13]

Dynamics of the Mass-Catcher

The payload stream, proceeding from the Moon to L_2 or to a target point P, is subject at launch to the effects of the Moon's geometrical librations, as well as to perturbations while in flight. If the mass-driver were to permit three degrees of freedom in control of the launch conditions, these being elevation, azimuth, and velocity, then there would be a range of launch latitudes from which it would be possible to reach L_2 with $C_{VT}=0$. Since the mass-driver is a large, fixed installation, there is effectively only one degree of freedom available to control the aim: launch velocity V_T. This control may be used to ensure that $C_{VT}=0$; the catcher then must traverse a periodic trajectory, centered at or near L_2, whose projection in the y-z plane permits the catcher to intercept the mass stream. In dealing with catcher dynamics, the principal effects are as follows:

1) Solar perturbations. Following developments due to Nicholson,[14] the first-order solar perturbation, relative to L_2, is

$$\Delta\ddot{r} = n''^2(\rho r_{me} - r_{mp}) \tag{2}$$

$\Delta\ddot{r} = \ddot{r}_p - \ddot{r}_{L2}$ and r_p = position of payload with respect to Earth, r_{L2} = position of L_2 with respect to Earth. n'' = solar mean motion = 0.0748, in normalized units; $\rho = 0.167833 = |r_{L2}| - 1$, r_{me} = position of Moon with respect to Earth, r_{mp} = position of payload with respect to the Moon. Hence, in 10^5 sec (the order of t_f), $|\Delta r| \sim 6$ km.

2) The perturbation in lunar orbit inclination. This has amplitude 10′23.7″ and period $32^d.279915$.[15] The corresponding motion in z has the same period and amplitude of some 200 km.

3) The libration in longitude. This has the amplitude, 7.9°; it is the geometrical effect associated with the eccentricity of the lunar orbit. We approximate the effect, for purposes of initial estimation, as leading to a change in launch longitude ϕ by ±7.9° or ±0.14 radians, which in view of the value of C_ϕ, (Fig. 4), gives the required catcher excursion in y as ±0.0231 distance units, or 8890 km. The linearized equations governing motion near L_2 then serve to find the control accelerations on the catcher needed for this:

$$\ddot{x} - 2\dot{y} = \Omega_{xx}x + A \sin t \quad ; \quad \ddot{y} + 2\dot{x} = \Omega_{yy}y + B \cos t \tag{3}$$

Here Ω_{xx}, Ω_{yy} are partial derivatives of Ω, Eq. (1), evaluated at L_2 and having values $\Omega_{xx} = 7.38084$, $\Omega_{yy} = -2.19042$; x, y are measured relative to L_2; t is normalized time. We wish to find the control acceleration amplitudes A, B such that (A^2+B^2) is minimized and $y = y_{max}\cos t$. There will also be $x = x_{max}\sin t$;

$$x_{max}/y_{max} = 2(2+\Omega_{xx}+\Omega_{yy})/[4+(\Omega_{xx}+1)^2] = 0.2046 \tag{4}$$

$$(A^2+B^2)^{1/2} = 1.621\ y_{max} = 0.0375 \tag{5}$$

where $(A^2+B^2)^{1/2}$ is in units of normalized acceleration and hence has the value, 1.02×10^{-4} m/sec^2.

4) The lunar obliquity. This has value, $\psi = 7.6°$ and represents the tilt of the lunar rotation axis with respect to the normal to the mean orbit plane. For an equatorial launch site, launching due east, the resulting perturbations at launch are

$$Z = r_m \sin\psi \sin t \quad ; \quad V_Z = V_T \sin\psi \cos t \tag{6}$$

where again r_m = lunar radius. Since at L_2 the linearized out-of-plane motion is given by, $\ddot{z} + \Omega_{zz}z = 0$, the requisite acceleration on the catcher is

$$\ddot{z}_c = (\Omega_{zz} - 1)\sin\psi\,[(r_m C_Z)^2 + (V_T C_{VZ})^2]^{1/2} \sin(t+t_f) \tag{7}$$

where Ω_{zz} at L_2 has value 3.19042. Expressing C_Z, C_{VZ} in normalized units, $C_Z = 33.2$, $C_{VZ} = 9 \times 10^{-3}$. Then the amplitude of $\ddot{z}_c$ is 0.039, very nearly equal that of the estimated amplitude of the in-plane control acceleration.

The orbit of the catcher then is a forced three-dimensional oscillation about L_2 wherein the projection on the y-z plane is an ellipse of semiaxes y_{max}, $[(r_m C_Z)^2 + (V_T C_{VZ})^2]^{1/2}\sin\psi$.

5) The perturbing acceleration due to the momentum flux from the incoming payload stream. This stream typically delivers 30 kg/sec at an arrival velocity $V_{L2} = 260$ m/sec, for a force of 8000 newtons. This gives an acceleration of 10^{-4} m/sec^2 on a catcher, nearly loaded, of mass 8×10^7 kg, a typical value.[3] If this acceleration is to be nulled by con-

tinuous thrusting, the propulsion requirements may exceed those required to cause the catcher to follow its nominal path. It is therefore of interest to consider how the catcher orbit may be offset so that this perturbing acceleration is nulled by the potential gradient whose components are Ω_x, Ω_y.

We thus define a modification of the restricted three-body problem:

$$\ddot{x}-2\dot{y}=\Omega_x+f_x(x,y,z;\ m;\ t) \tag{8a}$$

$$\ddot{y}+2\dot{x}=\Omega_y+f_y(x,y,z;\ m;\ t) \tag{8b}$$

$$\ddot{z}=\Omega_z+f_z(x,y,z;\ m;\ t) \tag{8c}$$

Here f_x, f_y, f_z represent a field of perturbing accelerations due to the payload stream arriving at (x,y,z), impacting a catcher of mass $m=m(t)$ and with the payload trajectory shapes being, in general, functions of the time-variable launch site longitude and hence of t. In the simplest case we consider m to be fixed and regard the Moon as of zero eccentricity (hence, libration in longitude) and obliquity. Then f_x, f_y, f_z represent constant biases on the acceleration and one may speak of a dynamical libration point.

Figure 7 gives the locus of the dynamical libration point associated with L_2. The x and y axes are the usual rotating coordinate system. Surrounding L_2 are arcs of contours of constant acceleration due to the potential gradient, $[\Omega_x^2+\Omega_y^2]^{1/2}$, which is zero at L_2. The dashed line extending to the right is the locus for which $\Omega_x=0$; below that line the potential gradient is in the direction of the Moon. The dashed line curving down and generally to the left is found as the locus of points along a family of trajectories integrated numerically from the Moon at $\phi=\phi_c=0.5777$, initially tangent to the surface and with varying V_T. Along such trajectories, it is found that there exist points where $\dot{x}/\dot{y}=\Omega_x/\Omega_y$, i.e. the local velocity vector is antiparallel to the potential gradient. Such points constitute the locus of dynamical libration points. Associated with each point on the locus is a value of $[\Omega_x+\Omega_y]^{1/2}$ which represents the fraction, (payload stream momentum flux)/(catcher mass).

Two integrated trajectories are shown: the nominal case, which passes through L_2, and a case for which V_T is increased by 0.06. The latter case is close to the limit for which increased V_T is required to permit catching at a dynamical libration point. As the catcher fills, it moves upward along the locus. The arrival velocity, nominally 260 m/sec when the catcher is at L_2, increases to 300 m/sec at the dynamical equilibria near $x=1.12$.

In the more general case where we require the catcher to execute motion in the vicinity of such dynamical equilibria so as to follow the target locus defined by Definition 1, the control accelerations are found from linearized equations which bear the same relation to Eqs. (8) as Eqs. (3) bear to Eqs. (1). There then is no net control acceleration needed to null out the momentum flux.

Colony Location and Transfer Orbits

Following the original suggestion of O'Neill,[1] most work to date has emphasized the concept of locating the colony at L_5 in the Earth-Moon system. This follows the work of Kolenkiewicz and Carpenter,[16] who found stable periodic orbits in the vicinity of L_5 (and L_4), through numerical integration of a restricted four-body problem wherein the motions of Earth, Moon and Sun corresponded to ephemerides. Such orbits thus possess the properties of long-term avoidance of close approaches to Earth or Moon and consequent freedom from strong gravity fields and from solar occultation.

However, L_5 and its vicinity have the disadvantage of having for the value of the Jacobi constant, $C_{L5}=3.0$. By contrast, at L_2, $C_{L2}=3.18408$. The difference represents a ΔV of 0.429 or 440 m/sec. Since this ΔV, or one of comparable magnitude, is to be applied to a total mass of material slightly greater than the Great Pyramid of Cheops,[17] it is clearly of interest to investigate colony locations for which this ΔV is reduced.

A site has been identified for which it is reduced by a factor of nearly 50. This is an orbit in 2:1 resonance with the Moon, i.e. with period approximately half that of the Moon, or two weeks. Such an orbit has the additional advantages that the ΔV required to reach the colony from low Earth orbit is reduced 5.3% (3873.8 vs 4091.2 m/sec); the transfer time from LEO is cut 50% (2.5 days vs 5.0 days); and, when transporting powersats from the colony to geosynchronous orbit, the transfer ΔV in low thrust is reduced approximately by 13.2% (1945 vs 2240 m/sec.)

In consideration of the 2:1 resonance in the plane, results pertinent to this paper have been given by Goldreich,[18] by Greenberg,[19] by Brouwer,[20] and by one of us.[21,22] Rather than use Eq. (1) of the restricted three-body problem to study such orbits, it is convenient to use the method of variation of parameters, developing the lunar perturbation as a disturbing function, in terms of orbital elements. This is done elsewhere,[15,22] with the result, to second order in the colony eccentricity e and lunar eccentricity e':

$$R=\mu[\tfrac{1}{2}b_{1/2}^{(0)}+T_1(e^2+e'^2)+T_2 e\cos\phi_e+T_3 e'\cos\phi_{e'} + T_4 ee'\cos(\phi_e-\phi_{e'})+T_5 e^2\cos 2\phi_e + T_6 ee'\cos(\phi_e+\phi_{e'}) + T_7 e'^2\cos 2\phi_{e'}] \tag{9a}$$

$$T_1=\frac{1}{8}(D+D^2)b_{1/2}^{(0)} \tag{9b}$$

$$T_2=-\frac{1}{2}(4+D)b_{1/2}^{(2)} \tag{9c}$$

$$T_3=\frac{1}{2}(3+D)b_{1/2}^{(1)} \tag{9d}$$

$$T_4=\frac{1}{4}(2-D-D^2)b_{1/2}^{(1)} \tag{9e}$$

$$T_5=\frac{1}{8}(44+13D+D^2)b_{1/2}^{(4)} \tag{9f}$$

$$T_6=-\frac{1}{4}(42+13D+D^2)b_{1/2}^{(3)} \tag{9g}$$

$$T_7=\frac{1}{8}(38+13D+D^2)b_{1/2}^{(2)} \tag{9h}$$

Here $b_{1/2}^{(j)}$ is a Laplace coefficient, found as a function of colony semimajor axis α by means of recursion relations.[15,22]

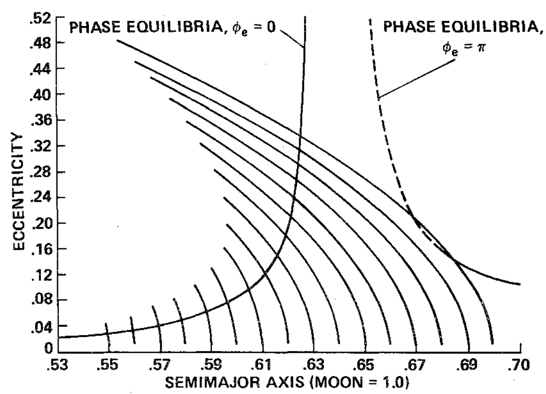

Fig. 8 Brouwer curves and phase equilibria for the lunar 2:1 resonance.

$Db_{1/2}^{(j)} = \alpha\, db_{1/2}^{(j)}/d\alpha$; $D^2 b_{1/2}^{(j)} = D(Db_{1/2}^{(j)})$. We have $e' = 0.0549$. The variables ϕ_e, $\phi_{e'}$ are given:

$$\phi_e = 2\lambda' - \lambda - \tilde{\omega} \quad ; \quad \phi_{e'} = 2\lambda' - \lambda - \tilde{\omega}' \tag{10}$$

λ, λ' are the mean longitudes of colony and Moon; by convention, $\tilde{\omega}' = 0$ and $\lambda' = t$ (time). In the absence of solar perturbations, $\tilde{\omega}' = \text{const}$. $\tilde{\omega}$, $\tilde{\omega}'$ are the arguments of the perigees of colony and Moon. The given disturbing function R is derived from the elliptic restricted three-body problem; the circular problem is associated with $e' = 0$. The variations of the orbital elements then are given by the Lagrange planetary equations,

$$\frac{d\alpha}{dt} = -\frac{2}{n\alpha}\left(\frac{\partial R}{\partial \phi_e} + \frac{\partial R}{\partial \phi_{e'}}\right) \tag{11a}$$

$$\frac{de}{dt} = \frac{(1-e^2)^{1/2}}{n\alpha^2 e}\left\{[2-(1-e^2)^{1/2}]\frac{\partial R}{\partial \phi_e} + [1-(1-e^2)^{1/2}]\frac{\partial R}{\partial \phi_{e'}}\right\} \tag{11b}$$

$$\frac{d\epsilon}{dt} = -\frac{2}{n\alpha}DR + \frac{(1-e^2)^{1/2}}{n\alpha^2 e}[1-(1-e^2)^{1/2}]\frac{\partial R}{\partial e} \tag{11c}$$

$$\frac{d\tilde{\omega}}{dt} = \frac{(1-e^2)^{1/2}}{n\alpha^2 e}\frac{\partial R}{\partial e} \tag{11d}$$

where n = colony mean motion. Also, ϵ is defined by: $\lambda = \int n\, dt + \epsilon$. Then,

$$d\phi_e/dt = 2 - n - d\tilde{\omega}/dt - d\epsilon/dt \; ; d\phi_{e'}/dt = 2 - n - d\epsilon/dt \tag{12}$$

There then are the following major results:

1) The variable ϕ_e, which defines the geometry of the three bodies at conjunction, may either librate (vary about the value $\phi_e = 0°$ or $\phi_e = 180°$ with amplitude $<180°$) or circulate (continually increase or decrease). In libration, the geometry of conjunction always prevents the colony, at apogee, from closely approaching the Moon. Indeed, at apogee the colony tends to be at quadrature with the Earth, away from the Moon. This is in consequence of a physical mechanism described by Goldreich[18] and Greenberg.[19] For small e, ϕ_e can librate about either 0° or 180°, but for large e, ϕ_e can librate only about 0°, corresponding to a colony apogee 90° out of phase with the position of the Moon.

2) There exist two quasi-integrals of the motion: the Jacobi integral, in the form due to Tisserand,

$$C = \frac{1}{\alpha} + 2[\alpha(1-e^2)]^{1/2} + R \tag{13}$$

and a quasi-integral due to Brouwer,[20] which when $e' = 0$ takes the form

$$B = \sqrt{\alpha}[2 - (1-e^2)^{1/2}] \tag{14}$$

3) When $e' = 0$ there exists, for any α, a value of e for which $d\alpha/dt = de/dt = d\phi_e/dt = 0$, the so-called phase equilibrium value:[21]

$$e_{pe} = \frac{\mu\alpha n\, T_2 \cos\phi_e}{2 - n - 2\mu n\alpha(T_1 + T_3 - 1/\alpha) - d\epsilon/dt} \tag{15}$$

where $\cos\phi_e = \pm 1$, being chosen in all cases such that $e_{pe} > 0$. Then, the orbit evolution in α, e, ϕ_e is periodic and may be regarded as an oscillation about a phase equilibrium, when in a libration with arbitrary initial conditions.

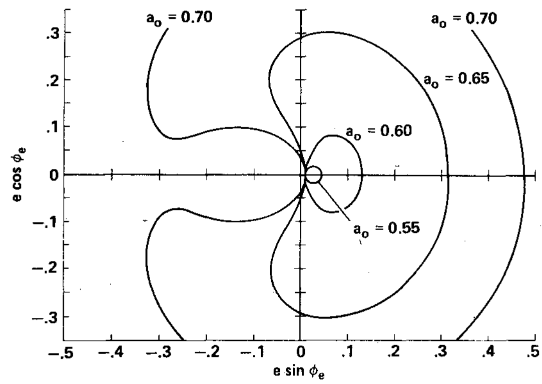

Fig. 9 Greenberg curves for the lunar 2:1 resonance.

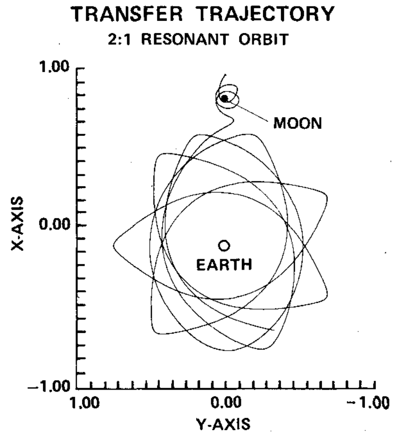

Fig. 10 Escape from L_2 into lunar orbit and subsequent Earth orbit.

Figure 8 illustrates variations in α, e, ϕ_e for typical orbits near the lunar 2:1 resonance (located at $\alpha = 0.62991$), with $e' = 0$. All such orbits were studied by integrating Eqs. (9-12) in single precision on Caltech's IBM 370-158, with initial values $e_0 = 0.01$, $(\phi_e)_0 = 0.0$ for each value of α ranging from 0.55 to 0.70, in jumps of 0.005. Each such orbit represents a curve of constant Brouwer integral, Eq. (14); the variations in α, e (Brouwer curves) are shown. Also shown are the curves of phase equilibrium. The latter curves are dashed in the region where apogee conjunctions are unstable, i.e. where conjunction with $\phi_e = 180°$ is not allowed by the Goldreich-Greenberg mechanism. The two branches of these curves correspond respectively to conjunctions with $\phi_e = 0°$, $\phi_e = 180°$.

Figure 9 illustrates shapes of typical variations in the e, ϕ_e domain (Greenberg curves); these curves are symmetric about $\sin\phi_e = 0$. Such curves were studied by Greenberg.[19] The labels on the curves refer to the starting values of α_0, with $e_0 = 0.01$.

Consider now the problem of transfer from L_2 to such an orbit. Let the catcher depart from L_2 with small velocity ($\dot{x} = -0.01$) in the direction of the Moon. The subsequent motion is shown in Fig. 10, found by integration of Eqs (1). The catcher executes three highly eccentric orbits about the Moon, then escapes lunar orbit through the vicinity of L_1. Such behavior is entirely consistent with that found in other studies of the theory of libration-point capture.[23] The time for transition from L_2 to L_1 is approximately 5.0 time units or 21.7 days. At escape, approximately, $\alpha = 0.57$, $e = 0.27$.

By continuing the integration, noting variation in α, e (found as functions of time directly from the computed variables x, y, $\dot{x}$, $\dot{y}$), it was possible to determine the condition of maximum α, minimum e, corresponding to $\sin\phi_e = 0$. This was reached at $t = 27.50$, with $\alpha = 0.6121634$, $e = 0.1521503$. It then was possible to match the indicated solution of Eqs. (1) to a solution found by integration of Eqs. (9-12). In this manner it was found that the given conditions correspond to $\phi_e = 180°$, so that the catcher orbit subsequent to escape through L_1 is in circulation and can make close approaches to the Moon. Hence it is not possible for the catcher orbit directly to evolve into a configuration such as those of Fig. 9. The catcher orbit may have the same value of B and C (or α and e) as a colony, but will be 180° opposite in phase along the physical orbit. Consequently, the catcher must exercise a transfer maneuver so as to rendezvous with the colony. The colony orbit then may be selected as that one of the type shown in Fig. 8, for which the transfer cost is minimized.

Figure 11 shows the Brouwer curve of the catcher orbit. The transfer consists of transition to a neighboring point of a curve of constant B, under the condition that this point correspond to a state of libration. The Brouwer curves of Fig. 9 approximate closely the maximum variations in α, e consistent with libration since $e_0 = 0.01$; $e_0 < 0$ would correspond to circulation. A near-optimal transfer involves a single impulse performed at perigee, increasing both α and e, and hence performed with $\phi_e = 0$ for the catcher. Hence the colony must also be at $\phi_e = 0$ with maximum e and minimum α. The increase in α, e due to the impulse ΔV is given by

$$\frac{\Delta V}{V} = \frac{1}{2}\frac{\Delta\alpha}{\alpha}\frac{1-e}{1+e}; \qquad \frac{\Delta e}{\Delta\alpha} = \frac{1-e}{\alpha} \tag{16}$$

where V = orbital velocity and the second of Eqs. (16) reflects the fact that the perigee altitude is unchanged by the impulse. Then, Fig. 11 also shows the selected colony orbit, and the transfer. The transfer is from the state ($\alpha = 0.5746$, $e = 0.2938$, $\phi_e = 0°$) to ($\alpha = 0.5894$, $e = 0.3127$, $\phi_e = 0°$). The latter state corresponds to a colony orbit with initial state $\alpha_0 = 0.65$, $e_0 = 0.01$. The transfer cost is $\Delta V = 0.0088 = 9.02$ m/sec, a 48-fold reduction over the transfer cost to L_5 inferred from the Jacobi constant. Fig. 12 shows the same orbits as Greenberg curves and confirms that the transition is indeed from circulation into libration, representing a so-called Type I capture into libration.[21]

The period of circulation in ϕ_e of the pre-transition catcher orbit is 25.94 or approximately eight times its Kepler period. Consequently, since the catcher is at $\phi_e = 180°$ at $t = 27.50$, it should be at $\phi_e = 0°$ at $t = 14.53$. This is confirmed by the integration of Eqs. (1), for which the peak in eccentricity is found at $t = 16.25$; then, $\alpha = 0.5621$, $e = 0.3681$. The difference between these values and those cited, e.g. $\alpha = 0.5746$, $e = 0.2938$, is due to the approximate nature of the resonance theory employed and is of no practical consequence. Consequently, it is concluded that the transfer time from L_2 to the colony is some 15 time units or 65 days. Under a maximum acceleration equal to that of Eq. (5), the catcher can execute the transfer ΔV, 9.02 m/sec, in 15.09 hours. This is small in comparison with the Kepler period (two weeks), thus confirming the impulsive nature of the transfer. The period of the libration in ϕ_e of the selected colony orbit is 43.6 (189 days), which gives the interval between opportunities for a transfer, L_2 to colony, of this type.

Of course, this launch window can be opened substantially by accepting somewhat higher values of transfer ΔV for the transition from the catcher orbit to colony orbit. The third Brouwer curve in Fig. 11 actually is the phase equilibrium at $\alpha = 0.6305$, $e = 0.35625$, $\phi_e = 0°$. Such a colony orbit is near-optimal in an operational sense since it has an infinite launch window; it can always be reached from the catcher orbit, with $\Delta V = 30.84$ m/sec, when the catcher is at $\phi_e = 0°$. For any libration about this equilibrium, it will always be true that for part of the libration, the transfer ΔV will be greater.

There remains for consideration the long-term stability of the colony orbit libration, in the presence of solar perturbations. These principally have the effect of secular perturbations in $\tilde{\omega}$, $\tilde{\omega}'$ and do not influence directly α, e, α' or

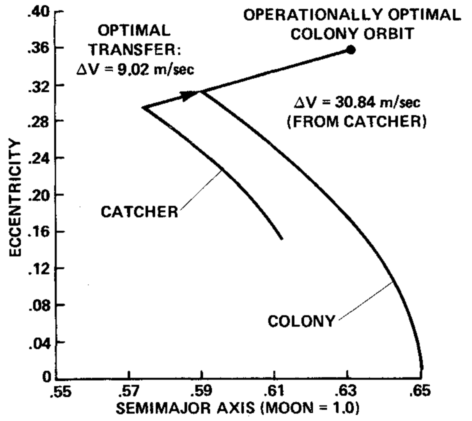

Fig. 11 Selection of an optimal colony orbit.

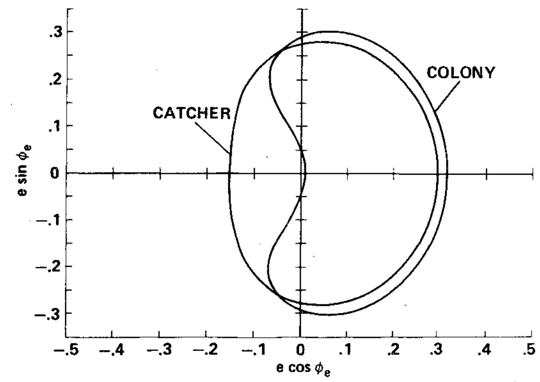

Fig. 12 Greenberg curves for the optimal transfer of Fig. 11.

e' to any significant extent. To be precise, α' (lunar semimajor axis) and e' undergo variations the largest terms of which are[15]

$$\delta\alpha' = n''^2\cos(2\lambda' - 2\lambda'') + \frac{15}{8}n''e'\cos(\lambda' - 2\lambda'' + \tilde{\omega}')$$

$$\delta e' = \frac{9}{4}n''^2\cos(\lambda' - 2\lambda'' + \tilde{\omega}') + \frac{1}{4}n''^2\cos(3\lambda' - 2\lambda'' - \tilde{\omega}')$$

$$+ \frac{15}{8}n''e'\cos(\lambda' - 2\lambda'' + \tilde{\omega}') \tag{17}$$

where n'' is as in Eq. (2) and λ'' = mean longitude of the Earth-Moon barycenter about the Sun. All the periods of variation in Eqs. (17) are of the order of a month. The corresponding expressions for the colony are found by writing α, e, λ, and n''/n for α', e', λ', n'' in Eqs. (17).

Of greater concern are the secular perturbations on $\tilde{\omega}$, $\tilde{\omega}'$. We have[15]

$$\frac{1}{n'}\frac{d\tilde{\omega}'}{dt} = \frac{3}{4}n''^2 + \frac{225}{32}n''^3 + \frac{4071}{128}n''^4 + \frac{265493}{2048}n''^5 + \ldots \tag{18}$$

where $n' = 1$. The corresponding expression for the colony is found by writing $\tilde{\omega}$, n, and n''/n for $\tilde{\omega}'$, $\tilde{\omega}'$, n', and n''. In addition, there are other secular terms

$$\frac{1}{n'}\frac{d\epsilon'}{dt} = -n''^2[1 + \frac{3}{2}(e'^2 + e''^2)]$$

$$\frac{1}{n}\frac{d\epsilon}{dt} = -\frac{n''^2}{n^2}[1 + \frac{3}{2}(e^2 + e''^2)] \tag{19}$$

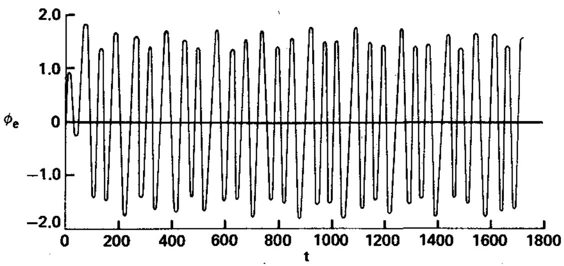

Fig. 13 Demonstration of long-term stability of the resonance variable ϕ_e in the presence of lunar eccentricity and solar perturbations.

and e' = Sun eccentricity = 0.0168. The resonance variables obey new equations:

$$\frac{d\phi_e}{dt} = 2[l + \frac{d\epsilon'}{dt}] - [n + (\frac{d\epsilon}{dt})_{11} + (\frac{d\epsilon}{dt})_{19}] - [(\frac{d\tilde{\omega}}{dt})_{11} + (\frac{d\tilde{\omega}}{dt})_{18}] \quad (20)$$

$$\frac{d\phi_e}{dt} = 2[l + \frac{d\epsilon'}{dt}] - [n + (\frac{d\epsilon}{dt})_{11} + (\frac{d\epsilon}{dt})_{19}] - \frac{d\tilde{\omega}'}{dt} \quad (21)$$

where the subscripts refer to resonance-induced perturbations, Eq. (11), or to solar-induced effects, Eqs. (18, 19).

Equations (9-11, 20) were integrated for 1720 time units or 20.5 years, on Caltech's IBM 370-158. Initial conditions were $\alpha_0 = 0.63$, $e_0 = 0.10$, $\phi_e = \phi_{e'} = 0$. Of particular concern was whether ϕ_e would continue to librate about the value 0°, indicating that the Goldreich-Greenberg mechanism would prevent any danger of a close approach of the colony to the Moon. Figure 13 shows the time history of ϕ_e, and it is evident that this is the case. Consequently, the stability of the colony orbit, in the presence of solar perturbations, may be regarded as established.

Conclusions

We have considered in some detail the problem of large-scale lunar mass transport. The most important results are the existence of critical points on the lunar surface, from which lunar-launched trajectories are insensitive to errors in launch velocity, and the existence of a colony orbit reachable from the catching point at L_2 with < 10 m/sec in ΔV.

The most important requirement for catcher onboard propulsion is seen to be associated with the forced periodic motion needed to track the variations in direction of a minimum-sensitivity payload stream, such variations being imposed by lunar obliquity and libration. Even this propulsion requirement, however, may be largely eliminated if a mass-driver can be built capable of adjustment in three degrees of freedom: azimuth and elevation, as well as launch velocity.

Nor need the catcher physically leave its station near L_2 for the purpose of delivery of its payload to a colony. It would be entirely adequate to collect its payload (lunar material) into a compact mass, to be fitted with onboard propulsion for stationkeeping to its nominal transfer orbit. Such an artificial asteroid, launched from L_2 onto a trajectory such as that of Fig. 10, would pass close to the colony and could be retrieved by a colony-based tug. An orbit through L_1 to L_2, entirely similar to that of Fig. 10, would suffice to return the propulsion unit to the catcher.

Our results thus are strongly supportive of the fundamental feasibility of space colonization, as well as of the use of lunar materials for large-scale space industrialization.

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